Subject CM2 April 2022 – Paper A Questions

1 A fund earns an annual rate of return i_t, with the rate of return in any year being independent of the rate in any other year. The distribution of $\log(1+i_t)$ is normal with parameters μ and σ^2.

The mean of i_t is 5% and the standard deviation is 3%.

(i) Calculate μ and σ^2. [3]

(ii) Calculate the probability that the fund return for any year is between 1% and 3%. [3]

(iii) Comment on your answer to part (ii). [1]

A sum of £10,000 is invested into the fund.

(iv) Calculate the probability that the accumulated value of the fund at the end of 3 years is less than £11,000. [2]

[Total 9]

2 (i) Define the term 'loss ratio' as used in the Bornhuetter-Ferguson method for estimating outstanding claim amounts. [1]

The run-off triangle below shows cumulative claims incurred on a portfolio of insurance policies.

	Development year		
Accident year	0	1	2
2017	864	1,011	1,072
2018	798	915	
2019	820		

Annual premiums written for accident year 2019 were 1,520 and the ultimate loss ratio is assumed to be 92.5%. Claims can be assumed to be fully run off by the end of development year 2.

(ii) Calculate the total claims arising from accidents in 2019, using the Bornhuetter-Ferguson method. [5]

1 year later, an unexpected event has resulted in higher claims than expected. The run-off triangle is now as shown below.

	Development year		
Accident year	0	1	2
2018	798	915	1,320
2019	820	1,412	
2020	1,016		

(iii) Calculate the revised total claims arising from accidents in 2019, using the Bornhuetter-Ferguson method. [3]

(iv) Discuss the implications of your answer to part (iii) for the insurance company. [3]
[Total 12]

3 Consider a share, S_t, and a derivative on the share with a value at time t of $f(t,S_t)$.

(i) Define, in your own words, what is represented by each of the following Greeks for this derivative:

(a) Delta

(b) Gamma

(c) Vega. [3]

Consider another share, A_t, which pays no dividends. The continuously compounded risk-free rate is r. Let K be the fair price at time 0 of a forward contract on A_t maturing at time T.

(ii) State the formula for K. [1]

Under the risk-neutral measure, Q, the share is expected to grow at the risk-free rate.

(iii) Demonstrate that the expected present value of the forward contract at time t $(0 \le t \le T)$ under the measure Q is $A_t - e^{-r(T-t)}K$. [2]

(iv) Calculate Delta, Gamma and Vega for the forward contract. [2]

(v) Comment on how the Greeks for the forward contract in part (iv) compare to the same Greeks for the underlying share. [2]

(vi) Discuss whether it would be appropriate to use a forward contract to Delta hedge a European call option on the share. [3]
[Total 13]

4 Suppose that under the unique equivalent measure martingale measure, Q, for a term structure model, the stochastic differential equation satisfied by the instantaneous interest rate r is:

$$dr_t = \alpha(\mu - r_t)dt + \sigma dZ_t$$

where $\alpha > 0$, μ and σ are fixed parameters and under Q, Z is a standard Brownian Motion.

The process X is defined by:

$$X_t = r_t b(T-t) + \int_0^t r_s ds$$

where the function b is given by $b(s) = \dfrac{(1-e^{-\alpha s})}{\alpha}$.

The function f is given by $f(x,t) = \exp(a(T-t) - x)$, where a is a differentiable function.

(i) Apply Ito's formula to $f(X_t, t)$. [6]

Hint: You may use, without proof, the fact that $dX_t = A_t dt + B_t dZ_t$ where $A_t = \alpha\mu b(T-t)$, and $B_t = \sigma b(T-t)$.

(ii) Find a differential equation that the function a must satisfy, in order for $f(X_t, t)$ to be a martingale. [2]

(iii) Determine an additional condition on a that is necessary for a bond with unit payoff at time T to have a price given by the formula:

$$B(t,T) = f(X_t, t)\exp\left(\int_0^t r_s ds\right)$$ [5]

[Total 13]

5 An insurance company holds a large amount of capital and wishes to distribute some to policyholders using one of two possible options.

Option A

A sum of £500 will be invested for each policyholder in a fund in which the expected annual effective rate of return is 3.5% and the standard deviation of annual returns is 2%. The annual rates of return are independent and $(1+i_t)$ is lognormally distributed with parameters μ and σ^2, where i_t is the rate of return in year t. The policyholder will receive the accumulated investment at the end of 15 years.

Option B

A sum of £500 will be invested for each policyholder for 10 years at a fixed rate of return of 4% *pa* effective. After 10 years, the accumulated sum will be invested for a further 5 years at the prevailing 5-year spot rate. This spot rate follows the probability density function shown below:

Spot rate (% *pa*)	Probability
0.5	0.15
1.0	0.25
4.5	0.40
7.0	0.20

The policyholder will receive the accumulated investment at the end of the 15 years.

(i) Demonstrate that $\mu = 0.0342$ and $\sigma = 0.0193$. [4]

(ii) Calculate the expected value and standard deviation at the end of year 15 of:

(a) Option A.

(b) Option B. [12]

(iii) Determine, for each of Options A and B, the probability that a policyholder's accumulated investment at the end of the 15 years will be less than £775. [5]

(iv) Compare the relative risk of the two options. [2]
[Total 23]

6 Consider a share with price S_0 at time $t=0$. The share will pay out a dividend of X at time $t=1$ and again at time $t=2$. The continuously compounded risk-free rate is r per unit of time.

Assume that the dividend payments are reinvested at the risk-free rate.

(i) Demonstrate that the fair price of a forward contract on S_t maturing at time $T>2$ is $K=(S_0-I)e^{rT}$, where I is the present value of the two dividends. [4]

A share is worth \$100 at time $t=0$. It will pay a dividend of \$5 at time $t=1$ and again at time $t=2$. The continuously compounded risk-free rate is 5% per unit of time.

(ii) Calculate the fair price of a forward contract on the share maturing at time $T=3$. [2]

An investor takes a long position in the forward contract in part (ii) at time $t=0$. Immediately afterwards, the proposed dividends on the underlying share are cancelled.

(iii) Discuss the implications of this for the investor. [2]
[Total 8]

7 An investor makes decisions based on the utility function $U(w) = w - 6w^2$, where w is the investor's wealth in millions of dollars ($\$m$).

(i) Demonstrate that the investor has both increasing absolute and relative risk aversion. [3]

The investor has $50,000 to invest over a 1-year period and has no other wealth. They have three options:

A Invest in a risk-free account. There will be no change in the value of the investment over 1 year.

B Invest in an asset that will give a 60% return over 1 year with probability 0.2, a 20% return with probability 0.7 and a –40% return with probability 0.1.

C Invest in an asset that will give a 30% return with probability 0.5 and a 20% return with probability 0.5.

The investor makes no allowance for discounting when making investment decisions. The investor must invest the whole $50,000 in a single option.

(ii) Determine which option the investor should choose to maximise their expected utility at the end of the year. [5]

(iii) Comment on why the investor could not use $U(w)$ to choose from the above options if their initial wealth was $65,000. [2]

[Total 10]

8 Consider a company funded entirely by debt and equity. The total value of the company's assets is $40 million. It has debt with a current outstanding amount of $20 million, with a continuously compounded interest rate of 8% *pa* and maturity in 7 years. Interest is added to the outstanding debt to be paid at maturity.

The volatility of the company's total assets is 10% *pa*. The continuously compounded risk-free rate of interest is 2% *pa*.

(i) Calculate the outstanding value of the debt, with interest, at maturity. [1]

(ii) Calculate the current value of the company's equity using the Merton model. [5]

(iii) Calculate the implied probability that the company will have sufficient assets to repay the debt at maturity. [1]

(iv) Demonstrate that the implied value of the company's debt is $29.7 million to the nearest $0.1 million (where the total value of the company is the value of the equity plus the value of the debt). [1]

(v) Explain why your answers to parts (i) and (iv) are different. [2]

An analyst wishes to set up a two-state credit model for the company. The two states will be denoted 'solvent' and 'default' with a constant transition intensity, λ, from solvent to default.

(vi) Calculate the value of λ that gives the same probability of default at maturity of the debt as calculated in part (iii). [2]

[Total 12]

END OF PAPER

Subject CM2 April 2022 – Paper A Solutions

Solution 1

Overview

This question is about the discrete-time lognormal model, which is covered in Chapter 9 of the Course Notes.

(i) ***Calculation of distribution parameters***

We know that the mean and standard deviation of i_t are given by:

$$E[i_t]=5\% \text{ and } sd(i_t)=3\%$$

Therefore we have:

$$E[1+i_t]=1.05$$ [½]

and:

$$Var(1+i_t)=Var(i_t)=(3\%)^2=0.0009$$ [½]

We also know that:

$$1+i_t \sim \text{lognormal}(\mu,\sigma^2)$$

From the *Formulae and Tables* we can say that for a lognormal distribution:

$$\begin{aligned} Var(1+i_t) &= (E[1+i_t])^2\left(e^{\sigma^2}-1\right) \\ &= 1.05^2\left(e^{\sigma^2}-1\right) \\ &= 0.0009 \end{aligned}$$ [½]

Rearranging gives:

$$\begin{aligned} \sigma^2 &= \log\left(1+\frac{0.0009}{1.05^2}\right) \\ &= (2.8566\%)^2 \end{aligned}$$ [½]

As noted in the Formulae and Tables, log *denotes logarithms to base e throughout these solutions.*

Again from the *Formulae and Tables* we have:

$$E[1+i_t]=e^{\mu+\frac{1}{2}\sigma^2}$$

$$=e^{\mu+\frac{1}{2}(2.8566\%)^2}$$

$$=1.05$$ [½]

Rearranging gives:

$$\mu=\log(1.05)-\frac{1}{2}(2.8566\%)^2$$

$$=4.8382\%$$ [½]

[Total 3]

(ii) ***Probability of annum return within range***

For this solution we need the following result for a random variable X:

$$P(a<X<b)=P(X<b)-P(X<a)$$

Without loss of generality, denote the annual return for any year as *i*. Then we have:

$$P(1\%<i<3\%)=P(i<3\%)-P(i<1\%)$$

$$=P(1+i<1.03)-P(1+i<1.01)$$

$$=P(\log(1+i)<\log(1.03))-P(\log(1+i)<\log(1.01))$$ [1]

But we know that $\log(1+i)\sim N(\mu,\sigma^2)$, so with the parameter values from part (i) we have:

$$P(1\%<i<3\%)=P\left(N\left(4.8382\%,(2.8566\%)^2\right)<\log(1.03)\right)$$

$$-P\left(N\left(4.8382\%,(2.8566\%)^2\right)<\log(1.01)\right)$$

$$=P\left(N(0,1)<\frac{\log(1.03)-4.8382\%}{2.8566\%}\right)$$

$$-P\left(N(0,1)<\frac{\log(1.01)-4.8382\%}{2.8566\%}\right)$$

$$=\Phi\left(\frac{\log(1.03)-4.8382\%}{2.8566\%}\right)-\Phi\left(\frac{\log(1.01)-4.8382\%}{2.8566\%}\right)$$

$$=\Phi(-0.65895)-\Phi(-1.3454)$$ [1]

We can use the NORM.S.DIST(x,TRUE) function in Excel to calculate the value of $\Phi(x)$.

Hence:

$$P(1\% < i < 3\%) = 0.25496 - 0.08925$$

$$\approx 16.6\%$$ [1]

[Total 3]

(iii) ***Comments on the probability***

The probability calculated in part (ii) is small. [½]

This is reasonable as the expected return in any year is 5%, and we are being asked to calculate the probability that the return is within a range which does not include the expected value. [½]

[Total 1]

Examiners confirmed that any other sensible comment on the magnitude of the probability would be awarded with [½].

(iv) ***Probability that fund accumulates to a minimum value***

Let S_n denote the accumulation of an initial unit investment at the end of *n* years. Therefore:

$$S_n = \prod_{t=1}^{n}(1+i_t)$$

$$\Rightarrow \qquad \log(S_n) = \sum_{t=1}^{n}\log(1+i_t)$$

Since $\log(1+i_t) \sim N\left(\mu,\sigma^2\right)$, then by independence we can say that:

$$\log(S_n) \sim N\left(n\mu, n\sigma^2\right)$$ [½]

The probability of an initial investment of £10,000 accumulating to less than £11,000 after three years is given by:

$$\begin{aligned} P(10{,}000S_3 < 11{,}000) &= P(S_3 < 1.1) \\ &= P(\log(S_3) < \log(1.1)) \\ &= P(N(3\mu, 3\sigma^2) < \log(1.1) \\ &= \Phi\left(\frac{\log(1.1) - 3\times 4.8382\%}{\sqrt{3}\times 2.8566\%}\right) \\ &= \Phi(-1.00726) \end{aligned}$$ [½]

We can use the NORM.S.DIST(x,TRUE) function in Excel to calculate the value of $\Phi(x)$.

Hence:

$$P(10{,}000S_3 < 11{,}000) \approx 15.7\%$$ [1]

[Total 2]

E[1 + i_t] = exp(mu + 0.5*sigma^2)

S_n = product_{t=1}^{n}(1 + i_t)

log(S_n) = sum_{t=1}^{n}log(1 + i_t)

P(10,000S_3 < 11,000) = PHI((log(1.1) – 3*4.8382%) / (sqrt(3) * 2.8566%))

Solution 2

Overview

The details of the Bornhuetter-Ferguson model required for this question can be found in Chapter 18 of the Course Notes.

The second half of the question presents an interesting scenario where the future doesn't turn out as originally expected. Discussing the consequences of this requires an understanding of the underlying run-off triangle principles as opposed to just being able to compute the numerical results.

(i) ***Definition of loss ratio***

A loss ratio is the ratio of incurred claims to earned premiums over a defined period. [1]

The loss ratio can be thought of as the proportion of the premiums that are expected to be spent paying claims.

(ii) ***Calculation of total claims arising under the Bornhuetter-Ferguson model***

With a loss ratio of 92.5% and premiums of 1,520 for accident year 2019, the initial estimate of ultimate loss for that year is given as:

$$1{,}520 \times 92.5\% = 1{,}406$$ [½]

The run-off triangle given shows the cumulative claims incurred, therefore the development factors can be calculated from it directly.

Let $DF_{i,j}$ denote the development factor from development year *i* to *j*. From the data given in the question we have:

$$DF_{1,2} = \frac{1,072}{1,011} = 1.0603 \qquad [1]$$

and:

$$DF_{0,1} = \frac{1,011+915}{864+798} = 1.1588 \qquad [1]$$

With an initial estimate of ultimate loss of 1,406, we would expect the claims paid in 2019 development year 0 to be:

$$\frac{1,406}{DF_{0,1} \times DF_{1,2}} = \frac{1,406}{1.1588 \times 1.0603} = 1,144.24 \qquad [½]$$

Therefore the emerging liabilities for accident year 2019 are:

$$1,406 - 1,144.24 = 261.76 \qquad [1]$$

The total claims are equal to what has already been paid in respect of the accidents in 2019, plus the liabilities that are expected to emerge from future claims.

Hence the total claims for accident year 2019 are:

$$820 + 261.76 = 1,081.76 \qquad [1]$$

[Total 5]

(iii) ***Revised total claims arising***

The method from part (ii) needs rerunning with different data, which is ideal if Excel has been used as the calculator.

With revised data we have a new development factor denoted by $DF^*_{1,2}$:

$$DF^*_{1,2} = \frac{1,320}{915} = 1.4426 \qquad [1]$$

With an (unchanged) initial estimate of ultimate loss of 1,406, we would expect the claims paid in 2019 development year 1 to be:

$$\frac{1,406}{DF^*_{1,2}} = \frac{1,406}{1.4426} = 974.61$$

Therefore the emerging liabilities for accident year 2019 are:

$$1,406 - 974.61 = 431.39 \qquad [1]$$

Hence the revised total claims for accident year 2019 are:

$$1,412 + 431.39 = 1843.39$$ [1]

[Total 3]

(iv) ***Implications for the insurance company***

Projected total claims for 2019 now exceed the premiums collected. [½]

This will result in a loss for the insurance company. [½]

It might make a loss on policies written in 2018… [½]

…and 2020 as well… [½]

…and possibly on earlier accident years that it thought were run off. [½]

It might need to revise its premiums… [½]

…and/or revise the expected loss ratio. [½]

This challenges all of the underlying assumptions of the model… [½]

…*eg* the need to take inflation into account. [½]

[Maximum 3]

Examiners confirmed that other valid comments would be awarded with [½].

Solution 3

Overview

The required definitions of the Greeks can be found in Chapter 11 of the Course Notes. This question also uses terminology and techniques from the introduction to derivative securities found in Chapter 10.

The question wording helpfully reminds us that under the risk-neutral probability measure risky assets are expected to grow at the risk-free rate. This is a central principle to derivative pricing and is critical for Subject CM2.

(i) ***Definitions of the Greeks***

(a) Delta is the rate of change of the value of the derivative with respect to changes in the price of the underlying. [1]

(b) Gamma is the rate of change of Delta with respect to changes in the price of the underlying. [1]

(c) Vega is the rate of change of the value of the derivative with respect to changes in the volatility of the underlying. [1]

[Total 3]

The question says that 'words' are required, therefore no marks are available for algebraic definitions of the Greek that aren't accompanied with an explanation.

(ii) ***Formula for the forward price***

The forward price, K, is given by the formula:

$$K = A_0 e^{rT}$$ [1]

(iii) ***Expected present value of the forward contract***

Under the risk-neutral probability measure, risky assets are expected to grow at the risk-free rate, therefore we have:

$$E_Q[A_T \mid F_t] = A_t e^{r(T-t)}$$

So the expected future value of the share price at time T, given what we know at time t, is the current share price A_t accumulated at the risk-free rate between times t and T.

Under Q, the present value of the forward contract is given by the general risk-neutral pricing formula as:

$$PV = e^{-r(T-t)} E_Q[A_T - K \mid F_t] \qquad [1]$$

where $A_T - K$ is the payoff function of the forward contract at expiry.

Since K is a constant and A_t is expected to grow at the risk-free rate we have:

$$PV = e^{-r(T-t)} \left(A_t e^{r(T-t)} - K \right) \qquad [½]$$

Simplifying gives the required expression:

$$PV = A_t - Ke^{-r(T-t)} \qquad [½]$$

[Total 2]

The 'value' of the forward is not the same thing as the 'price' of the forward contract, as we shall see in Question 6.

(iv) ***Greeks for the forward contract***

Starting with the formula for the value of the forward contract from part (iii) we can take partial derivatives to find the Greeks:

$$Delta = \frac{\partial}{\partial A_t}\left(A_t - Ke^{-r(T-t)} \right) = 1 \qquad [1]$$

$$Gamma = \frac{\partial}{\partial A_t}\left(Delta \right) = 0 \qquad [½]$$

$$Vega = \frac{\partial}{\partial \sigma}\left(A_t - Ke^{-r(T-t)} \right) = 0 \qquad [½]$$

[Total 2]

For Vega it is tempting to try and differentiate A_t with respect to σ – after all, σ is the volatility of the share price and so there must be some connection between the two. However, when taking partial derivatives it is necessary to only vary one parameter at a time, whilst assuming that all others remain constant. The sensitivity to changes in the share price is captured by the Delta, and therefore A_t can be held constant whilst determining Vega.

(v) ***Comparing the Greeks for the forward contract and underlying share***

The Greeks of the underlying share are given by:

$$Delta = \frac{\partial}{\partial A_t}\left(A_t\right) = 1$$

$$Gamma = \frac{\partial}{\partial A_t}\left(Delta\right) = 0$$

$$Vega = \frac{\partial}{\partial \sigma}\left(A_t\right) = 0$$ [1]

These are clearly the same as those of the forward contract in part (iii). [½]

This is because a forward is effectively the same as owning the share if it pays no dividends. [½]
[Total 2]

(vi) ***Delta-hedging a European call with a forward contract***

In theory, a forward contract could be used to Delta-hedge a European call… [1]

…by shorting a number of forwards equal to the Delta of the call. [1]

However, there would be no control of the Gamma of the hedge over time. [½]

This would make it likely that regular rebalancing is required… [½]

…as the Delta of the call would be changing regularly… [½]

…but the Delta of the 'hedging' forwards would not change with them. [½]

This could be costly in terms of time and resources. [½]

The hedge would also have no control of the Vega over time. [½]

This would mean that the hedger would be more exposed to changes in the volatility of the asset. [½]
[Maximum 3]

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K = A_0 * exp(r*T)

PV = exp(-r * (T-t)) * E_Q[A_T – K | F_t]

Gamma = @Delta/@A_t, where @ indicates a partial derivative
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Solution 4

Overview

The Vasicek model for the short rate of interest is covered in Chapter 15 of the Course Notes.

Some of the results and proofs required here are not explicitly dealt with in the Core Reading, which demonstrates the importance of understanding the model and an appreciation for what it is trying to achieve.

Part (iii) is arguably the hardest question in the paper. Fortunately, general mathematical reasoning rather than knowledge of complex Subject CM2 content would be sufficient to deal with it.

(i) ***Applying Ito's Lemma***

Using the notation from the *Formulae and Tables*, we can identify the Ito process as:

$$dx = a(x,t)dt + b(x,t)dz$$

where:

$$x = X_t$$

$$z = Z_t$$

$$a(x,t) = A_t$$

$$b(x,t) = B_t \qquad [1]$$

In this solution there are two functions called 'a' and two functions called 'b', this arises because of the overlapping notation used in the question and the Formulae and Tables. The number of arguments in each function can be used to determine their source. For example, $b(x,t)$ is the function defined in the Formulae and Tables whereas $b(T-t)$ is specified by the question wording.

To apply Ito's Lemma to the function $f(x,t) = \exp(a(T-t) - x)$ we need to know its partial derivatives:

$$\frac{\partial}{\partial x} f(x,t) = -\exp(a(T-t) - x) = -f(x,t) \qquad [½]$$

$$\frac{\partial^2}{\partial x^2} f(x,t) = f(x,t) \qquad [½]$$

$$\frac{\partial}{\partial t} f(x,t) = -a'(T-t)\exp(a(T-t) - x) = -a'(T-t) f(x,t) \qquad [1]$$

where:

$$a'(T-t)=\frac{\partial}{\partial t}a(T-t)$$

Substituting these into the formula for dG in the *Formulae and Tables*, where $G=f(x,t)$, gives:

$$df(x,t)=\left(A_t\frac{\partial f(x,t)}{\partial x}+\frac{1}{2}B_t^2\frac{\partial^2 f(x,t)}{\partial x^2}+\frac{\partial f(x,t)}{\partial t}\right)dt+B_t\frac{\partial f(x,t)}{\partial x}dz \qquad [1]$$

$$=\left(-A_t f(x,t)+\frac{1}{2}B_t^2 f(x,t)-a'(T-t)f(x,t)\right)dt-B_t f(x,t)dZ_t \qquad [1]$$

Simplifying gives:

$$df(x,t)=f(x,t)\left(-A_t+\frac{1}{2}B_t^2-a'(T-t)\right)dt-B_t f(x,t)dZ_t \qquad [1]$$

But:

$$A_t=\alpha\mu b(T-t) \text{ and } B_t=\sigma b(T-t)$$

So we have:

$$df(x,t)=f(x,t)\left(-\alpha\mu b(T-t)+\frac{1}{2}\sigma^2 b^2(T-t)-a'(T-t)\right)dt-\sigma b(T-t)f(x,t)dZ_t \qquad [1]$$

Given that $b(s)=\dfrac{1-e^{-\alpha s}}{\alpha}$ we can write:

$$df(x,t)=f(x,t)\left(-\mu\left(1-e^{-\alpha(T-t)}\right)+\frac{\sigma^2}{2\alpha^2}\left(1-e^{-\alpha(T-t)}\right)^2-a'(T-t)\right)dt$$

$$-\frac{\sigma}{\alpha}\left(1-e^{-\alpha(T-t)}\right)f(x,t)dZ_t \qquad [1]$$

[Maximum 6]

Alternatively we could use Taylor's formula (even though the question wording asks for Ito's formula).

$$df(X_t,t)=\frac{\partial f(X_t,t)}{\partial X_t}dX_t+\frac{1}{2}\frac{\partial^2 f(X_t,t)}{\partial X_t^2}(dX_t)^2+\frac{\partial f(X_t,t)}{\partial t}dt$$

$$=-f(X_t,t)dX_t+\frac{1}{2}f(X_t,t)(dX_t)^2-a'(T-t)f(x,t)dt$$

$$=f(X_t,t)\left(-A_t dt-B_t dZ_t+\frac{1}{2}(A_t dt+B_t dZ_t)^2-a'(T-t)dt\right)$$

$$=f(X_t,t)\left(-A_t+\frac{1}{2}B_t^2-a'(T-t)\right)dt-f(X_t,t)B_t dZ_t$$

$$=f(X_t,t)\left(-\alpha\mu b(T-t)+\frac{1}{2}\sigma^2 b^2(T-t)-a'(T-t)\right)dt-f(X_t,t)\sigma b(T-t)dZ_t$$

$$=f(x,t)\left(-\mu\left(1-e^{-\alpha(T-t)}\right)+\frac{\sigma^2}{2\alpha^2}\left(1-e^{-\alpha(T-t)}\right)^2-a'(T-t)\right)dt$$

$$-\frac{\sigma}{\alpha}\left(1-e^{-\alpha(T-t)}\right)f(x,t)dZ_t$$

(ii) ***Martingale condition***

A necessary condition for $f(X_t,t)$ to be a martingale is that the process must have a zero drift term in the stochastic differential equation that describes its behaviour. Therefore the dt coefficient in the expression for $df(X_t,t)$ must equal zero. [1]

From part (i) we can see that this condition is satisfied by the following differential equation:

$$-\mu\left(1-e^{-\alpha(T-t)}\right)+\frac{\sigma^2}{2\alpha^2}\left(1-e^{-\alpha(T-t)}\right)^2-a'(T-t)=0 \quad [1]$$

[Total 2]

This can also be expressed as:

$$-A_t+\frac{1}{2}B_t^2-a'(T-t)=0$$

(iii) ***Bond price formula***

It may not be immediately obvious now to proceed with this question, but a clue is given in the question in that the bond pays out 1 at time T. This is essentially setting the boundary condition for the proposed pricing formula. Once this is established the remainder of the question requires general mathematical reasoning.

The only time we know the price of the bond with certainty is when it matures at time *T*. Therefore in order for the bond pricing formula $B(t,T)$ to be correct at maturity we require the boundary condition $B(T,T)=1$ to be met, *ie*: [1]

$$B(T,T)=f(X_T,T)\exp\left(\int_0^T r_s ds\right)=1 \qquad [½]$$

Rearranging then gives:

$$f(X_T,T)=\exp\left(-\int_0^T r_s ds\right) \qquad [½]$$

From the definition of $f(X_T,T)$ we have:

$$\exp(a(T-T)-X_T)=\exp\left(-\int_0^T r_s ds\right) \qquad [½]$$

$$\Rightarrow \qquad a(0)=X_T-\int_0^T r_s ds \qquad [1]$$

Then from the definition of X_T we have:

$$\begin{aligned} a(0)&=r_T b(T-T)+\int_0^T r_s ds-\int_0^T r_s ds \\ &=r_T b(0) \end{aligned} \qquad [½]$$

But:

$$\begin{aligned} b(0)&=\frac{1-e^{-\alpha\times 0}}{\alpha} \\ &=\frac{1-1}{\alpha} \\ &=0 \end{aligned}$$

Therefore the additional condition on the function *a* is that $a(0)=0$. [1]

[Total 5]

dx = a(x,t) * dt + b(x,t) * dz

@f(x,t)/@t = –a'(T-t) * exp(a(T-t) – x), where @ indicates a partial derivative

df(x,t) = (A_t * @f(x,t)/@x + 0.5 * B_t^2 * @^2f(x,t)/@x^2 + @f(x,t)/@t)dt + B_t * @f(x,t)/@x * dz

B(T,T) = f(X_T,T) * exp(integral_{0}^{T} r_s * ds)

b(0) = (1 – exp(–alpha * 0)) / alpha

Solution 5

Overview

This long question covers the same material as Question 1. The discrete-time lognormal model to help us, which is covered in Chapter 9 of the Course Notes.

If Excel was used for Question 1 then those earlier calculations can easily be recycled here.

(i) ***Calculation of distribution parameters***

Whilst these calculations are identical to those found in Question 1 part (i), they are here worth four marks rather than three.

From the question we know that the mean and standard deviation of i_t are given by:

$$E[i_t] = 3.5\% \quad \text{and} \quad sd(i_t) = 2\%$$

Therefore we have:

$$E[1+i_t] = 1.035$$ [½]

and $$Var(1+i_t) = Var(i_t) = (2\%)^2 = 0.0004$$ [½]

We also know that:

$$1+i_t \sim \text{lognormal}(\mu, \sigma^2)$$

From the *Formulae and Tables* we can say that:

$$Var(1+i_t) = (E[1+i_t])^2 \left(e^{\sigma^2} - 1\right)$$

$$= 1.035^2 \left(e^{\sigma^2} - 1\right)$$

$$= 0.0004$$ [1]

Rearranging gives:

$$\sigma^2 = \log\left(1+\frac{0.0004}{1.035^2}\right)$$

$$= (1.9322\%)^2 \qquad [½]$$

Again from the *Formulae and Tables* we can say that:

$$E[1+i_t] = e^{\mu+\frac{1}{2}\sigma^2}$$

$$= e^{\mu+\frac{1}{2}(1.9322\%)^2}$$

$$= 1.035 \qquad [1]$$

Rearranging gives:

$$\mu = \log(1.035) - \frac{1}{2}(1.9322\%)^2$$

$$= 3.42\% \qquad [½]$$

[Total 4]

(ii) ***Expected value and standard deviation of accumulated investment***

Working in £'s, let A and B be random variables to denote the accumulated investment after 15 years if an initial deposit of 500 is invested according to Option A and Option B respectively.

(ii)(a) ***Option A***

Under Option A we have:

$$A = 500\prod_{t=1}^{15}(1+i_t) \qquad [1]$$

$$\Rightarrow \qquad \log(A) = \log(500) + \sum_{t=1}^{15}\log(1+i_t) \qquad [½]$$

We know that $\log(1+i_t) \sim N\left(\mu,\sigma^2\right)$ and that each annual return is independent. [1]

Therefore we can say that:

$$\log(A) \sim N\left(\log(500)+15\mu, 15\sigma^2\right) \qquad [1]$$

Using the values of μ and σ from part (i) we have:

$$\log(A) \sim N\left(\log(500)+15\times 3.42\%, 15\times(1.9322\%)^2\right) \qquad [½]$$

From the *Formulae and Tables* we find:

$$E[A] = \exp\left(\log(500) + 15 \times 3.42\% + \tfrac{1}{2} 15 \times (1.9322\%)^2\right)$$

$$= 837.67 \qquad [1½]$$

And

$$Var(A) = 837.67^2 \left(\exp\left(15 \times (1.9322\%)^2\right) - 1\right)$$

$$= 3940.55 \qquad [1½]$$

Therefore:

$$sd(A) = \sqrt{Var(A)}$$

$$= 62.77 \qquad [½]$$

(ii)(b) ***Option B***

Under Option B we have:

$$B = 500(1+4\%)^{10}(1+r)^5$$

$$= 740.12(1+r)^5 \qquad [1]$$

where *r* is the prevailing spot rate after ten years.

From the data in the question we have:

$$E[B] = 740.12 \times E\left[(1+r)^5\right]$$

$$= 740.12\left(1.005^5 \times 0.15 + 1.01^5 \times 0.25 + 1.045^5 \times 0.4 + 1.07^5 \times 0.2\right)$$

$$= 884.83 \qquad [1]$$

The variance of *B* can then be calculated via:

$$Var(B) = E[B^2] - E^2[B]$$

$$= 740.12^2 \times E\left[(1+r)^{2\times 5}\right] - 884.83^2 \qquad [1]$$

$$= 740.12^2\left(1.005^{2\times 5} \times 0.15 + 1.01^{2\times 5} \times 0.25 + 1.045^{2\times 5} \times 0.4 + 1.07^{2\times 5} \times 0.2\right)$$

$$-884.83^2$$

$$= 10{,}500.18 \qquad [1]$$

Therefore:

$$sd(B) = \sqrt{Var(B)}$$

$$= 102.47$$ [½]

[Total 12]

(iii) ***Probability of accumulating to less than £775***

For Option A, the probability of the £500 initial investment accumulating to less than £775 after 15 years is given by:

$$P(A < 775) = P\left(\log(A) < \log(775)\right)$$ [1]

We know that:

$$\log(A) \sim N\left(\log(500) + 15 \times 3.42\%, 15 \times (1.9322\%)^2\right)$$ [½]

Therefore:

$$P(A < 775) = P\left(N\left(\log(500) + 15 \times 3.42\%, 15 \times (1.9322\%)^2\right) < \log(775)\right)$$ [½]

Standardising gives:

$$P(A < 775) = \Phi\left(\frac{\log(775/500) - 15 \times 3.42\%}{\sqrt{15} \times 1.9322\%}\right)$$

$$= \Phi(-1.00178)$$

$$= 15.8\%$$ [1]

For Option B, the possible accumulation amounts are:

- $740.12 \times 1.005^5 = 758.81$ with probability 0.15 [½]
- $740.12 \times 1.01^5 = 777.88$ with probability 0.25 [½]
- $740.12 \times 1.045^5 = 922.33$ with probability 0.4
- $740.12 \times 1.07^5 = 1{,}038.06$ with probability 0.2

There is only one outcome for which $B < 775$, and this occurs with a probability of 0.15. Therefore:

$$P(B < 775) = 15\%$$ [1]

[Total 5]

(iv) ***Comparison of the relative risks***

Option A is riskier in terms of a lower minimum possible payout… [1]

…but Option B is riskier in terms of having a larger standard deviation. [1]

The risk of a shortfall relative to £775 is greater for Option A. [1]

[Maximum 2]

Var(1 + i_t) = (E[1 + i_t])^2 * (exp(sigma^2) – 1)

Var(B) = E[B^2] – (E[B])^2

Solution 6

Overview

This question on forward contracts uses material covered in Chapter 10 of the Course Notes.

An important point to remember when dealing with forward contracts is that the 'forward price' (also known as the 'price of the forward') is not the same thing as the 'value of the forward'. The forward price is the price at which the underlying asset will be traded at a specified future maturity date.

(i) ***Demonstrate the fair price of the forward contract***

We will follow the 'two portfolio' argument here which requires two portfolios to have matching cashflows at future time T. Some creativity is required to generate these portfolios, but since one of them must contain the forward contract we could also include with it enough cash to be able to execute that trade in the future.

Consider Portfolio 1 with the following components:

- one forward contract
- Ke^{-rT} in cash where K is the price of the forward. [½]

At time T, Portfolio 1 will be worth:

$$\underbrace{S_T - K}_{\text{payoff from the forward contract}} + \underbrace{Ke^{-rT}e^{rT}}_{\text{accumulation of the cash component}} = S_T$$ [½]

Now we need to find another portfolio which is also worth S_T at time T. The underlying share itself looks like a good candidate, but we must factor in the receipt of dividends.

Consider Portfolio 2 with the following components:

- one share (which pays dividends)
- $-I$ in cash where I is the present value of the two dividends. [½]

At time $T>2$ both of the dividends will have been paid, and so Portfolio 2 will be worth:

$$\underbrace{S_T + Ie^{rT}}_{\substack{\text{the share plus}\\ \text{the accumulated}\\ \text{present value}\\ \text{of the dividends}}} - \underbrace{Ie^{rT}}_{\substack{\text{accumulation}\\ \text{of the cash}\\ \text{component}}} = S_T$$ [½]

Using the no-arbitrage argument, given that both portfolios are guaranteed to have the same value at time T, they must also be worth the same at all earlier times. [1]

Specifically at time $t=0$ we have:

$$Ke^{-rT} = S_0 - I$$ [½]

The value of the forward contract is initially zero.

Rearranging gives:

$$K = (S_0 - I)e^{rT}$$ [½]

[Total 4]

Since we have been asked to demonstrate the fair price, it would also be possible to examine the two cases where $K>(S_0-I)e^{rT}$ and $K<(S_0-I)e^{rT}$ and then search for a contradiction (by finding an arbitrage opportunity). This proof by contradiction would lead to the conclusion that $K=(S_0-I)e^{rT}$.

(ii) ***Fair price of the forward contract***

Working in dollars, the present value of the two dividends, I, is given by:

$$I = 5e^{-5\%\times 1} + 5e^{-5\%\times 2}$$
$$= 9.28$$ [1]

From the result in part (i) we have:

$$K = (100 - 9.28)e^{5\%\times 3}$$
$$= 105.40$$ [½]

Therefore the fair price of the forward contract is \$105.40. [½]

[Total 2]

(iii) ***Implications for the investor***

In isolation, the new forward price will be higher without any dividends… [½]

…but the investor has already contracted to buy the share for the old, lower forward price. [½]

So the investor's long position in the forward contract will now have a positive value. [½]

The investor might also consider why the dividends have been cancelled. [½]

If the share price has also moved because of the announcement then this will also change the value of the investor's position. [½]

[Maximum 2]

S_T – K + K *exp(–rt) * exp(rt) = S_T

K = (S_0 – I) * exp(rt)

Solution 7

Overview

The utility theory required for this question can be found in Chapter 2 of the Course Notes.

Parts (i) and (ii) are standard proofs and calculations for this topic.

Part (iii) questions the validity of the quadratic utility function, which is always determined by the function's ability to reflect the key properties of investors; non-satiation and risk-aversion.

(i) ***Increasing risk aversion***

Let $A(w)$ and $R(w)$ denote absolute and relative risk aversion respectively.

From the *Formulae and Tables* we have:

$$A(w) = -\frac{U''(w)}{U'(w)} \text{ and } R(w) = w \times A(w)$$

Since $U(w) = w - 6w^2$, the derivatives are given by:

$$U'(w) = 1 - 12w \text{ and } U''(w) = -12$$ [½]

Therefore:

$$A(w) = \frac{12}{1-12w}$$ [½]

Differentiating with respect to *w* gives:

$$A'(w)=\frac{d}{dw}12(1-12w)^{-1}$$

$$=144(1-12w)^{-2}$$

$$=\frac{144}{(1-12w)^2}>0$$ [½]

Therefore $A(w)$ is an increasing function of wealth *w*. [½]

Since we have shown that $A(w)$ is increasing, and *w* increases with wealth by definition, then the product $w\times A(w)$ is also an increasing function. [½]

Hence $R(w)=w\times A(w)$ is an increasing function of wealth. [½]

[Total 3]

We could instead differentiate $R(w)$ with respect to w to give:

$$R'(w)=\frac{d}{dw}(w\times A(w))$$

$$=\frac{d}{dw}12w(1-12w)^{-1}$$

$$=12(1-12w)^{-1}+144w(1-12w)^{-2}$$

$$=\frac{12(1-12w)+144w}{(1-12w)^2}$$

$$=\frac{12}{(1-12w)^2}>0$$

(ii) ***Investment strategy to maximise expected utility***

The utility function is defined in terms of w where the wealth is expressed in terms of millions of dollars. So in this part of the question the investor has an initial wealth of $w=0.05$.

Let *A*, *B* and *C* denote the utilities achieved by the investor when investing fully in the respective options.

Option A

$$E[A]=U(0.05)$$

$$=0.05-6\times 0.05^2$$

$$=0.035$$ [1]

Option B

Under this option the final wealth will be:

- $0.05 \times (1+60\%) = 0.08$ with probability 0.2
- $0.05 \times (1+20\%) = 0.06$ with probability 0.7
- $0.05 \times (1-40\%) = 0.03$ with probability 0.1 [½]

Therefore the expected utility is given by:

$$E[B] = U(0.08) \times 0.2 + U(0.06) \times 0.7 + U(0.03) \times 0.1$$

$$= (0.08 - 6 \times 0.08^2) \times 0.2 + (0.06 - 6 \times 0.06^2) \times 0.7 + (0.03 - 6 \times 0.03^2) \times 0.1$$

$$= 0.03766$$ [1]

Option C

Under this option the final wealth will be:

- $0.05 \times (1+30\%) = 0.065$ with probability 0.5
- $0.05 \times (1+20\%) = 0.06$ with probability 0.5 [½]

Therefore the expected utility is given by:

$$E[C] = U(0.065) \times 0.5 + U(0.06) \times 0.5$$

$$= (0.065 - 6 \times 0.065^2) \times 0.5 + (0.06 - 6 \times 0.06^2) \times 0.5$$

$$= 0.039$$ [1]

The investor should choose the option which maximises their expected utility. Since $E[C] > E[B] > E[A]$ the investor should choose Option C. [1]

[Total 5]

(iii) ***Comments on the validity of the utility function***

Utility functions need to represent the key properties of investors; non-satiation and risk-aversion. [½]

These two properties are imposed upon the utility function $U(w)$ by specifying that $U'(w) > 0$ and $U''(w) < 0$ respectively. [½]

With $U(w) = w - 6w^2$ it is clear that $U''(w) = -12 < 0$ for all w, so we should focus on the non-satiation requirement instead.

The given quadratic utility function is only valid for values of *w* such that:

$$U'(w) = 1 - 12w > 0$$

ie for values of $w < \frac{1}{12}$. This equates to a wealth level of $83,333. [½]

If the investor has an initial wealth of $65,000, then the threshold level of $83,333 is attained with an investment return of:

$$\frac{83,333}{65,000} - 1 = 28.2\%$$ [½]

Both options B and C offer a potential return in excess of 28.2%, and therefore the given utility function is not adequate to assess the investor's expected utility. [½]

[Maximum 2]

A(w) = –U''(w) / U'(w)

U'(w) = 1 – 12 * w > 0

Solution 8

Overview

The Merton model and the two-state model used in this question can be found in Chapter 16 of the Course Notes.

Parts (i) to (v) require standard Merton model calculations and results. It is crucial that you can quickly and accurately calculate the values of European options using the Black-Scholes formulae.

Part (v) links two credit risk models together via their probabilities of default.

(i) ***Value of the debt at maturity***

Working in millions of dollars, the value of the debt now is 20, so with a continuously compounded interest rate of 8% *pa* this will accumulate over the seven years to maturity to a value of:

$$20 \times e^{8\% \times 7} = 35.013$$

Therefore the value of the debt at maturity is $35.013*m*. [1]

(ii) ***Value of the company's equity***

Under the Merton model, the value of the company's equity is equal to that of a European call option written on the total assets of the company. The strike price of the option is equal to the value of the debt at maturity. [½]

Using that Black-Scholes pricing formula to calculate the value of this European call option, *c*, we have:

$$d_1 = \frac{\log\left(S/K\right) + \left(r + \tfrac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$= \frac{\log\left(40/35.013\right) + \left(2\% + \tfrac{1}{2}(10\%)^2\right)\times 7}{10\% \times \sqrt{7}}$$

$$= 1.1647 \qquad [1]$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$= 1.1647 - 10\% \times \sqrt{7}$$

$$= 0.9001 \qquad [1]$$

Then:

$$\Phi(d_1) = 0.8779 \qquad [½]$$

$$\Phi(d_2) = 0.8160 \qquad [½]$$

Combining these results gives:

$$c = S\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

$$= 40 \times 0.8779 - 35.013e^{2\% \times 7} \times 0.8160$$

$$= 10.28 \qquad [1]$$

Therefore the value of the company's equity is £10.28*m*. [½]

[Total 5]

(iii) ***Implied probability of successfully repaying the debt***

Given the number of marks available it is intended that the following Black-Scholes result is quoted without proof:

$$P(S_T > K) = \Phi(d_2)$$

The probability of successfully repaying the debt is equal to the probability of the company's assets exceeding the value of the debt after seven years:

$$\Phi(d_2) = 0.8160 \qquad [1]$$

(iv) ***Current value of the company's debt***

The Merton model states that the current total value of the company's assets is equal to the sum of the equity value and the current debt value. Therefore the value of the debt is given by:

$$40 - 10.28 = 29.72$$

To the nearest \$0.1*m* the debt has a value of \$29.7*m*. [1]

(v) ***Reasons why the debt values are different***

The value is the outstanding amount, rolled up to maturity at 8% *pa*, adjusted to allow for defaults, and discounted back at 2% *pa*. [1]

So the values differ because the debt has a chance of default and has a higher interest rate than the risk-free rate. [1]
[Total 2]

(vi) ***Calculating an equivalent default intensity***

Under the two-state model, the probability of default between time 0 and time T is given by:

$$1 - \exp\left(-\int_0^T \lambda(s)ds\right)$$

This expression takes the same form as the survival function in the Formulae and Tables:

$$_t p_x = \exp\left(-\int_0^t \mu_{x+s}ds\right)$$

When the transition intensity $\lambda(s) \equiv \lambda$ is constant this simplifies to:

$$1 - \exp(-\lambda T)$$

Therefore probability of *not* defaulting is $\exp(-\lambda T)$. Equating this to the result from part (iii) gives:

$$\exp(-\lambda \times 7) = 0.8160$$ [1]

Rearranging gives:

$$\lambda = -\frac{\log(0.8160)}{7}$$
$$= 0.0291$$ [1]
[Total 2]

```
d1 = (log(S/K) + ( r + 0.5*sigma^2) * T) / (sigma * sqrt(T))

d2 = d1 – sigma * sqrt(T)

c = S * PHI(d1) – K * exp(–rt) * PHI(d2)

exp(–int_{0}^{T} lambda(s) * ds ) = PHI(d2)
```

ActEd's Hints

Question 1

(i) The *Formulae and Tables* gives the required formulae for the mean and variance of the lognormal distribution.

(ii) Try using the following identity:

$$P(a < X < b) = P(X < b) - P(X < a)$$

(iii) With very little information to work with, consider the magnitude of the result from part (ii).

(iv) Find the distribution of the accumulated fund at the end of three years, then calculate how likely this is to be less than £11,000.

Question 2

(i) Bookwork definition.

(ii) The Bornhuetter-Ferguson method determines the initial estimate of the ultimate liability by combining the loss ratio with the premiums received. This initial estimate is then refined in the light of the development factors obtained from the claims data.

(iii) Repeat the calculations from part (ii) with different data.

(iv) The risk of insolvency for the insurer is always greater when the actual claims turn out to the greater than those expected.

Question 3

(i) Bookwork definition.

(ii) Bookwork definition.

(iii) Use the general risk-neutral pricing formula and the fact that forward contracts have zero value at inception.

(iv) Partially differentiate the forward contract pricing formula with respect to the various parameters.

(v) Partially differentiate the share price with respect to the various parameters and compare those to the results of part (iv).

(vi) A Delta-hedge is intended to remove or reduce the uncertainty around the value of the contract as the underlying share price moves. Focus on the deltas of the forward contract and the European call option.

Question 4

(i) Standard application of Ito's Lemma, or equivalently, Taylor's theorem.

(ii) Martingales are random processes which have zero drift. The result from part (i) can be interpreted as the combination of a drift term and a noise term.

(iii) Follow the clue in the question and determine the constraint on *a* such that the boundary condition $B(T,T)=1$ holds.

Question 5

(i) Follow the same method as Question 1 (i).

(ii) Under Option A the accumulated fund after 15 years will be distributed continuously as lognormal$(15\mu, 15\sigma^2)$, whereas under Option B the distribution will be discrete.

(iii) Use the distributions from part (ii) to examine the likelihood of the accumulated funds being less than £775.

(iv) Consider the list of measures of investment risk from the Course Notes and determine which ones are likely to be applicable here.

Question 6

(i) The 'fair price of a forward' is not the value of the forward, but the price at which the asset will be traded in the future. Construct two portfolios (one of which containing a forward contract) such that they are guaranteed to have the same value at time $T>2$, then apply the no-arbitrage argument.

(ii) Apply the formula given in part (i).

(iii) Dividends are only paid to the holder of the share and not the holder of the forward contract.

Question 7

(i) The formulae for the absolute and relative aversion to risk measures are found in the *Formulae and Tables*.

(ii) The expected utility can be calculated via:

$$E[U]=\sum_{i=1}^{n} p_i \times U(w_i)$$

(iii) Quadratic utility functions have a particular flaw which means that they are not applicable for all wealth levels.

Question 8

(i) Apply standard Merton model calculations where the value of the company's equity is equated to the value of a European call option written on the total assets of the company.

(ii) See hint to previous part.

(iii) In the Merton model context the probability of having sufficient funds to pay the debt at maturity is the same as the share price exceeding the strike price at time T in the Black-Scholes setting.

(iv) The company's debt plus equity value (as calculated in part (ii)) must equal the total assets.

(v) Consider the differences between the present value and the face value of a future debt repayment where the interest and discount rates are different.

(vi) Apply the survival probability formula from the two-state credit risk model.